# On the classification of some classes of doppelsemigroups up to isomorphism

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A *doppelsemigroup* [5,8] is an algebraic structure $(D, \dashv, \vdash)$ consisting of a non-empty set $D$ equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$$(D_1) \quad (x \dashv y) \vdash z = x \dashv (y \vdash z),$$

$$(D_2) \quad (x \vdash y) \dashv z = x \vdash (y \dashv z).$$

A doppelsemigroup $(D, \dashv, \vdash)$ is called *commutative* [6] if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative. A doppelsemigroup $(D, \dashv, \vdash)$ is said to be *strong* [7] if it satisfies the axiom

$$x \dashv (y \vdash z) = x \vdash (y \dashv z).$$

Each semigroup $(S, \dashv)$ can be consider as a (strong) doppelsemigroup $(S, \dashv, \dashv)$, and we denote this *trivial* doppelsemigroup by $S$. Let $(D, \dashv, \vdash)$ be a doppelsemigroup. Denote by $(D, \dashv, \vdash)^d$ its *dual* doppelsemigroup $(D, \dashv^d, \vdash^d)$, where $x \dashv^d y = y \dashv x$ and $x \vdash^d y = y \vdash x$. It follows that $(D, \dashv, \vdash)^d$ is a (strong) doppelsemigroup if and only if $(D, \dashv, \vdash)$ is a (strong) doppelsemigroup. Therefore, non-commutative doppelsemigroups are divided into the pairs of dual doppelsemigroups.

One of the main tasks in the study of algebraic structures is their classification up to isomorphism. In [3], the task of describing all pairwise non-isomorphic (strong) doppelsemigroups of order 3 has been solved.

**Theorem 1.** *There exist* 75 *pairwise non-isomorphic three-element doppelsemigroups among which* 41 *doppelsemigroups are commutative. Non-commutative doppelsemigroups are divided into* 17 *pairs of dual doppelsemigroups. Also up to isomorphism there are* 65 *strong doppelsemigroups of order* 3*, and all non-strong doppelsemigroups are not commutative. There exist exactly* 24 *pairwise non-isomorphic three-element trivial doppelsemigroups.*

A doppelsemigroup $(G, \dashv, \vdash)$ is called *a group doppelsemigroup* if $(G, \dashv)$ is a group. It follows that $(G, \vdash)$ is a group isomorphic to $(G, \dashv)$, see [1,2].

**Proposition 1.** *Let $(G, \dashv)$ be a group and let $(G, \dashv, \vdash)$ be a doppelsemigroup. Then $\vdash = \dashv_a$ for some $a \in G$.*

**Corollary 1.** *Let $(G, \dashv)$ be an Abelian group and let $(G, \dashv, \vdash)$ be a doppelsemigroup. Then $(G, \dashv, \vdash)$ is a commutative strong doppelsemigroup and $\vdash = \dashv_a$ for some $a \in G$.*

Recall that the number of divisors function $\tau(n)$ is defined as the number of natural divisors of a natural number $n$.

A doppelsemigroup $(G, \dashv, \vdash)$ said to be *cyclic* if $(G, \dashv)$ is a cyclic group. The following theorems on the number of pairwise non-isomorphic cyclic doppelsemigroups were proved by the author, see [4].

**Theorem 2.** *There exist $\tau(n)$ pairwise non-isomorphic finite cyclic doppelsemigroups of order $n$. All finite cyclic doppelsemigroups are strong and commutative.*

**Theorem 3.** *Let $a, b \in \mathbb{Z}$. The doppelsemigroups $(\mathbb{Z}, +, +_a)$ and $(\mathbb{Z}, +, +_b)$ are isomorphic if and only if $b \in \{a, -a\}$. Consequently, there exist countably infinite many pairwise non-isomorphic infinite cyclic (commutative strong) doppelsemigroups.*

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