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RESEARCH ARTICLE

Algebra in superextensions of semilattices

Taras Banakh[1] and Volodymyr Gavrylkiv

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Abstract. Given a semilattice X we study the algebraic properties of the semigroup $\upsilon(X)$ of upfamilies on X. The semigroup $\upsilon(X)$ contains the Stone-Čech extension $\beta(X)$, the superextension $\lambda(X)$, and the space of filters $\varphi(X)$ on X as closed subsemigroups. We prove that $\upsilon(X)$ is a semilattice iff $\lambda(X)$ is a semilattice iff $\varphi(X)$ is a semilattice iff the semilattice X is finite and linearly ordered. We prove that the semigroup $\beta(X)$ is a band if and only if X has no infinite antichains, and the semigroup $\lambda(X)$ is commutative if and only if X is a bush with finite branches.

Introduction

One of the most powerful tools in the modern Combinatorics of Numbers is the method of ultrafilters based on the fact that each (associative) binary operation $*: X\times X\to X$ defined on a discrete topological space X extends to a right-topological (associative) operation $*:\beta(X)\times\beta(X)\to\beta(X)$ on the Stone-Čech compactification $\beta(X)$ of X, see [9], [11]. The Stone-Čech extension $\beta(X)$ is the space of ultrafilters on X. The extension of the operation from X to $\beta(X)$ can be defined by the simple formula:

$$\mathcal{U}*\mathcal{V}=\Big\langle \bigcup_{x\in U} x*V_x : U\in\mathcal{U},\ \ (V_x)_{x\in U}\in\mathcal{V}^U\Big\rangle, \tag{1}$$

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where $\langle\mathcal{B}\rangle = \{A \subset X : \exists B \in \mathcal{B}\ \ B \subset A\}$ is the upper closure of a family $\mathcal{B}$. In this case $\mathcal{B}$ is called a *base* of $\langle\mathcal{B}\rangle$. Identifying each point $x \in X$ with the principal ultrafilter $\langle x\rangle = \{U \subset X : x \in U\}$, we identify X with a subspace of the Stone-Čech compactification.

Endowed with this extended operation, the Stone-Čech compactification $\beta(X)$ becomes a compact right-topological semigroup. The algebraic properties of this semigroup (for example, the existence of idempotents or minimal left ideals) have important consequences in combinatorics of numbers, see [9], [11].

In [8] it was observed that the binary operation $*$ extends not only to $\beta(X)$ but also to the space $\upsilon(X)$ of all upfamilies on X. By definition, a family $\mathcal{F}$ of non-empty subsets of a discrete space X is called an *upfamily* if for any sets $A \subset B \subset X$ the inclusion $A \in \mathcal{F}$ implies $B \in \mathcal{F}$. The space $\upsilon(X)$ is a closed subspace of the double power-set $\mathcal{P}(\mathcal{P}(X))$ endowed with the compact Hausdorff topology of the Tychonoff power $\{0,1\}^{\mathcal{P}(X)}$. In the papers [7], [8], [1]–[3] the space $\upsilon(X)$ was denoted by $G(X)$ and its elements were called inclusion hyperspaces[2]. The extension of a binary operation $*$ from X to $\upsilon(X)$ can be defined in the same way as for ultrafilters, i.e., by the formula (1) applied to any two upfamilies $\mathcal{U}, \mathcal{V} \in \upsilon(X)$. If X is a semigroup, then $\upsilon(X)$ is a compact Hausdorff right-topological semigroup containing $\beta(X)$ as a closed subsemigroup. The algebraic properties of this semigroups were studied in detail in [8].

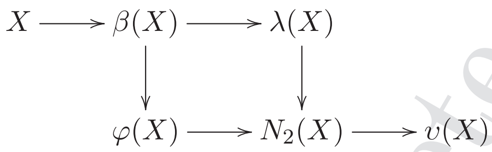
The space $\upsilon(X)$ of upfamilies over a discrete space X contains many interesting subspaces. First we recall some definitions. An upfamily $\mathcal{A} \in \upsilon(X)$ is defined to be

- *a filter* if $A_1 \cap A_2 \in \mathcal{A}$ for all sets $A_1, A_2 \in \mathcal{A}$;
- *an ultrafilter* if $\mathcal{A} = \mathcal{A}'$ for any filter $\mathcal{A}' \in \upsilon(X)$ containing $\mathcal{A}$;
- *linked* if $A \cap B \neq \emptyset$ for any sets $A, B \in \mathcal{A}$;
- *maximal linked* if $\mathcal{A} = \mathcal{A}'$ for any linked upfamily $\mathcal{A}' \in \upsilon(X)$ containing $\mathcal{A}$.

By $\varphi(X)$, $\beta(X)$, $N_2(X)$, and $\lambda(X)$ we denote the subspaces of $\upsilon(X)$ consisting of filters, ultrafilters, linked upfamilies, and maximal linked upfamilies, respectively. The space $\lambda(X)$ is called *the superextension* of X, see [10], [14]. In [8] it was observed that for a discrete semigroup X

[2] We decided to change the terminology and notation after discovering the paper [12, 2.7.4] that discusses monadic properties of the up-set functor υ.

the subspaces $\varphi(X)$, $\beta(X)$, $N_2(X)$, $\lambda(X)$ are closed subsemigroups of the semigroup $\upsilon(X)$. The following diagram describes the inclusion relations between these subspaces of $\upsilon(X)$ (an arrow $A \to B$ indicates that A is a subset of B).



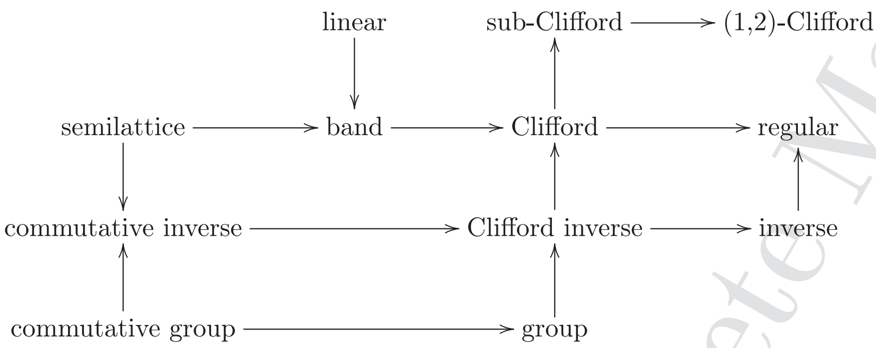
In [1] — [4] we studied the properties of the compact right-topological semigroup $\upsilon(X)$ and its subsemigroups for groups X. In this paper we shall study the algebraic structure of the semigroups $\lambda(X)$, $\varphi(X)$, $N_2(X)$, and $\upsilon(X)$ for semilattices X.

Let us recall that a *semilattice* is a commutative idempotent semigroup. Idempotent semigroups are called *bands*. So, in a band each element x is an *idempotent*, which means that $xx = x$. A semigroup S is *linear* if $xy \in \{x, y\}$ for any elements $x, y \in X$. It follows that each linear semigroup S is a band. Each (linear) semilattice is partially (linearly) ordered by the relation $\leq$ defined by $x \leq y$ iff $xy = x$.

A semigroup S is *cancellative* if for each element $a \in S$ the left shift $l_a : S \to S$, $l_a : x \mapsto ax$, and the right shift $r_a : S \to S$, $r_a : x \mapsto xa$, are injective. A semigroup S is called *Clifford* (resp. *sub-Clifford*) if S is a union of groups (resp. of cancellative semigroups). Observe that a subsemigroup of a sub-Clifford semigroup is sub-Clifford and a finite semigroup S is Clifford if and only if it is sub-Clifford. It is easy to see that a semigroup S is sub-Clifford if and only if for every pair of natural numbers n, m it is (n, m)-*Clifford* in the sense that for any element $x \in S$ the equality $x^{n+1} = x^{m+1}$ implies $x^n = x^m$.

A semigroup S is called *a regular semigroup* if $a \in aSa$ for any $a \in S$. Such a semigroup S is called *an inverse semigroup* if $ab = ba$ for any idempotents $a, b \in S$. Observe that each band is a Clifford semigroup and every Clifford semigroup is sub-Clifford and regular. An inverse semigroup with a unique idempotent is a group.

These algebraic properties of semigroups relate as follows:



In this paper we shall characterize semigroups X whose extensions $\upsilon(X)$, $\lambda(X)$, $\varphi(X)$ or $N_2(X)$ are bands, linear semigroups, commutative semigroups, or semilattices. In Section 5 we shall characterize lattices X whose extensions $\upsilon(X)$, $\lambda(X)$, $\varphi(X)$ are lattices. The results obtained in this paper will be applied in the paper [5] devoted to the superextensions of inverse semigroups.

1. Semigroups whose extensions are bands

In this section we shall characterize semigroups X whose extensions $\upsilon(X)$, $\lambda(X)$ or $\varphi(X)$ are bands. Let us recall that a semigroup S is a (linear) band if $xx = x$ for all $x \in X$ (and $xy \in \{x, y\}$ for all $x, y \in X$).

Let us recall that an element a of a semigroup S is *regular* in S if $a \in aSa$. It is clear that each idempotent is a regular element.

Theorem 1.1. *For a semigroup X the following conditions are equivalent:*

(1) *X is linear;*

(2) *$\upsilon(X)$ is a band;*

(3) *$\varphi(X)$ is a band;*

(4) *$\lambda(X)$ is a band.*

Proof. (1) ⇒ (2) Assume that the semigroup X is linear. To show that $\upsilon(X)$ is a band, we should check that $\mathcal{A}*\mathcal{A} = \mathcal{A}$ for any upfamily $\mathcal{A} \in \upsilon(X)$. Since X is linear, for any $A \in \mathcal{A}$ we get $A = A * A \in \mathcal{A} * \mathcal{A}$ and hence $\mathcal{A} \subset \mathcal{A} * \mathcal{A}$.

To show that $\mathcal{A} \supset \mathcal{A} * \mathcal{A}$, fix any basic subset $B = \bigcup_{x \in A} x{*}A_x \in \mathcal{A} * \mathcal{A}$ where $A \in \mathcal{A}$ and $A_x \in \mathcal{A}$ for all $x \in A$.

Now we consider two cases.

(i) There is $x \in A$ such that $xa = a$ for all $a \in A_x$. In this case $\mathcal{A} \ni A_x = x{*}A_x \subset B$ and thus $B \in \mathcal{A}$.

(ii) For every $x \in A$ there is $a \in A_x$ such that $xa \neq a$ and hence $xa = x$ (as X is linear). In this case $\mathcal{A} \ni A \subset \bigcup_{x \in A} x * A_x = B$ and hence $B \in \mathcal{A}$.

The implications $(2) \Rightarrow (3, 4)$ are trivial.

$(3) \Rightarrow (1)$ Assume that $\varphi(X)$ is a band. Then X, being a subsemigroup of $\varphi(X)$, also is a band. To show that X is linear, take any two points $x, y \in X$ and consider the filter $\mathcal{F} = \langle \{x, y\} \rangle \in \varphi(X)$. Being an idempotent, the filter $\mathcal{F}$ is a regular element of the semigroup $\upsilon(X)$. Consequently, we can find an upfamily $\mathcal{A} \in \upsilon(X)$ such that $\mathcal{F} * \mathcal{A} * \mathcal{F} = \mathcal{F}$. It follows that there are sets $A_x, A_y \in \mathcal{A}$ such that $(xA_x \cup yA_y) \cdot \{x, y\} \subset \{x, y\}$. In particular, for every $a_x \in A_x$ we get $xa_xy \in \{x, y\}$. If $xa_xy = x$, then $xy = xa_xyy = xa_xy = x$. If $xa_xy = y$, then $xy = xxa_xy = xa_xy = y$, witnessing that the band X is linear.

$(4) \Rightarrow (1)$ Assume that $\lambda(X)$ is a band. Then X, being a subsemigroup of $\lambda(X)$, is a band as well. Assuming that the band X is not linear, we can find two points $x, y \in X$ such that $xy \notin \{x, y\}$. We claim that the maximal linked system $\mathcal{L} = \langle \{x, y\}, \{x, xy\}, \{y, xy\} \rangle \in \lambda(X)$ is not an idempotent. We shall prove more: the element $\mathcal{L}$ is not regular in the semigroup $\upsilon(X)$. Assuming the converse, we can find an upfamily $\mathcal{A} \in \upsilon(X)$ such that $\mathcal{L} * \mathcal{A} * \mathcal{L} = \mathcal{L}$. It follows from $\{x, y\} \in \mathcal{L} = \mathcal{L} * \mathcal{A} * \mathcal{L}$ that $\{x, y\} \supset \bigcup_{u \in L} u * B_u$ for some set $L \in \mathcal{L}$ and some sets $B_u \in \mathcal{A} * \mathcal{L}$, $u \in L$. The linked property of the family $\mathcal{L}$ implies that the intersection $L \cap \{x, xy\}$ contains some point u. Now for the set $B_u \in \mathcal{A} * \mathcal{L}$ find a set $A \in \mathcal{A}$ and a family $(L_a)_{a \in A} \in \mathcal{L}^A$ such that $B_u \supset \bigcup_{a \in A} a * L_a$. Fix any point $a \in A$ and a point $v \in L_a \cap \{y, xy\}$. Then $uav \in uaL_a \subset uB_u \subset \{x, y\}$. Since $u \in \{x, xy\}$ and $v \in \{y, xy\}$, the element uav is equal to xby for some element $b \in \{a, ya, ax, yax\}$. So, $xby \in \{x, y\}$. If $xby = x$, then $xy = xbyy = xby = x \in \{x, y\}$. If $xby = y$, then $xy = xxby = xby = y \in \{x, y\}$. In both cases we obtain a contradiction with the choice of the points $x, y \notin \{x, y\}$. $\square$

Observe that the proof of Theorem 1.1 yields a bit more, namely:

Proposition 1.2. *For a band X the following conditions are equivalent:*

(1) *X is linear;*

(2) *each element of $\varphi(X)$ is regular in $\upsilon(X)$;*

(3) *each element of $\lambda(X)$ is regular in $\upsilon(X)$.*

The linearity of a semilattice X can be also characterized via the $(1,2)$-Clifford property of the semigroups $\varphi(X)$ and $\lambda(X)$.

Theorem 1.3. *For a semilattice X the following conditions are equivalent:*

(1) *X is linear;*

(2) *$\varphi(X)$ is $(1,2)$-Clifford;*

(3) *$\lambda(X)$ is $(1,2)$-Clifford.*

Proof. The implications $(1) \Rightarrow (2,3)$ follow from Theorem 1.1 because each band is a $(1,2)$-Clifford semigroup.

$(2,3) \Rightarrow (1)$ Assume that the semilattice X is not linear. Then X contains two elements $x, y \in X$ such that $yx = xy \notin \{x,y\}$.

Consider the filter $\mathcal{F} = \langle\{x,y\}\rangle$ and observe that $\mathcal{F} \neq \mathcal{F} * \mathcal{F} = \langle\{x, xy, y\}\rangle = \mathcal{F} * \mathcal{F} * \mathcal{F}$, which means that the semigroup $\varphi(X)$ is not $(1,2)$-Clifford.

To see that the semigroup $\lambda(X)$ is not $(1,2)$-Clifford, consider the maximal linked system $\mathcal{L} = \langle\{x,y\},\{x,xy\},\{y,xy\}\rangle \in \lambda(X)$ and observe that $\mathcal{L} \neq \mathcal{L} * \mathcal{L} = \langle\{xy\}\rangle = \mathcal{L} * \mathcal{L} * \mathcal{L}$. □

Next we characterize semigroups X whose Stone-Čech extension $\beta(X)$ is a band. A sequence $(x_n)_{n\in\omega}$ of points of some set X is called *injective* if $x_n \neq x_m$ for any distinct numbers $n, m \in \omega$.

Theorem 1.4. *For a band X the semigroup $\beta(X)$ is a band if and only if for each injective sequence $(x_n)_{n\in\omega}$ in X there are numbers $n < m$ such that $x_n x_m \in \{x_n, x_m\}$.*

Proof. To prove the "only if" part, assume that $(x_n)_{n\in\omega}$ is an injective sequence in X such that $x_n x_m \notin \{x_n, x_m\}$ for all $n < m$. We claim that there is an infinite subset $\Omega \subset \omega$ such that $x_n x_m \neq x_k$ for any numbers $n, m, k \in \Omega$ with $n < m$. For this we shall apply the famous Ramsey

Theorem. Consider the 4-coloring $\chi : [\omega]^3 \to 4 = \{0,1,2,3\}$ of the set $[\omega]^3 = \{(k,n,m) \in \omega^3 : k < n < m\}$, defined by

$$\chi(k,n,m) = \begin{cases} 1 & \text{if } x_k x_n = x_m, \\ 2 & \text{if } x_k x_m = x_n, \\ 3 & \text{if } x_n x_m = x_k, \\ 0 & \text{otherwise.} \end{cases}$$

By the Ramsey Theorem [11, 5.1], there is an infinite set $\Omega \subset \omega$ such that $\chi(\Omega^3 \cap [\omega]^3)$ is a singleton. It follows from the definition of the coloring Ω that this singleton is $\{0\}$, which means that for any numbers $k,n,m \in \Omega$ with $n < m$ and $k \notin \{n,m\}$ we get $x_n x_m \neq x_k$. Since $x_n x_m \notin \{x_n, x_m\}$ for any numbers $n < m$, we conclude that $x_n x_m \neq x_k$ for any numbers $k,n,m \in \Omega$ with $n < m$.

Now take any free ultrafilter $\mathcal{A}$ that contains the set $A = \{x_n\}_{n\in\Omega}$. Then for every $n \in \omega$ the set $A_{>n} = \{x_m : n < m \in \Omega\}$ belongs to the ultrafilter $\mathcal{A}$. The choice of the sequence $A = \{x_n\}_{n\in\Omega}$ guarantees that $A \cap \bigcup_{n\in\Omega} x_n * A_{>n} = \emptyset$, which implies that $\mathcal{A} \neq \mathcal{A} * \mathcal{A}$ and hence the ultrafilter $\mathcal{A}$ is not an idempotent in $\beta(X)$.

To prove the "if" part, assume that $\beta(X)$ is not a band and find an ultrafilter $\mathcal{F} \in \beta(X)$ with $\mathcal{F} * \mathcal{F} \neq \mathcal{F}$. In particular, $\mathcal{F} * \mathcal{F} \not\subset \mathcal{F}$. This implies that for some $A \in \mathcal{F}$ and $\{A_x\}_{x\in A} \subset \mathcal{F}$ the set $\bigcup_{x\in A} x*A_x \notin \mathcal{F}$.

Consider the set $X^{\uparrow}_{\mathcal{F}} = \{x \in X : {\uparrow}x \in \mathcal{F}\}$ where ${\uparrow}x = \{y \in X : xy = x\}$. We claim that $X^{\uparrow}_{\mathcal{F}} \notin \mathcal{F}$. Assuming that $X^{\uparrow}_{\mathcal{F}} \in \mathcal{F}$, we conclude that $A \cap X^{\uparrow}_{\mathcal{F}} \in \mathcal{F}$. This implies that ${\uparrow}a \in \mathcal{F}$ and ${\uparrow}a \cap A_a \in \mathcal{F}$ for any $a \in A \cap X^{\uparrow}_{\mathcal{F}}$. Therefore $a * ({\uparrow}a \cap A_a) = \{a\}$ and hence

$$\bigcup_{x\in A} x * A_x \supset \bigcup_{x\in A\cap X^{\uparrow}_{\mathcal{F}}} x * ({\uparrow}x \cap A_x) = \bigcup_{x\in A\cap X^{\uparrow}_{\mathcal{F}}} \{x\} = A \cap X^{\uparrow}_{\mathcal{F}} \in \mathcal{F}.$$

Thus $\bigcup_{x\in A} x*A_x \in \mathcal{F}$. This contradiction shows that $X^{\uparrow}_{\mathcal{F}} \notin \mathcal{F}$.

Next, consider the set $X^{\downarrow}_{\mathcal{F}} = \{x \in X : {\downarrow}x \in \mathcal{F}\}$ where ${\downarrow}x = \{y \in X : xy = y\}$. We claim that $X^{\downarrow}_{\mathcal{F}} \notin \mathcal{F}$. Assume that $X^{\downarrow}_{\mathcal{F}} \in \mathcal{F}$. Then $A \cap X^{\downarrow}_{\mathcal{F}} \in \mathcal{F}$. This implies that ${\downarrow}a \in \mathcal{F}$ and ${\downarrow}a \cap A_a \in \mathcal{F}$ for any $a \in A \cap X^{\downarrow}_{\mathcal{F}}$. Therefore

$${\downarrow}a \cap A_a \subset a * ({\downarrow}a \cap A_a) \subset a * A_a \subset \bigcup_{x\in A} x * A_x$$

and $\bigcup_{x\in A} x*A_x \in \mathcal{F}$. This contradiction shows that $X^{\downarrow}_{\mathcal{F}} \notin \mathcal{F}$.

Since $\mathcal{F}$ is an ultrafilter, $X^{\uparrow}_{\mathcal{F}} \cup X^{\downarrow}_{\mathcal{F}} \notin \mathcal{F}$ and $Z_{\mathcal{F}} = X \setminus (X^{\uparrow}_{\mathcal{F}} \cup X^{\downarrow}_{\mathcal{F}}) \in \mathcal{F}$. Let $x_0 \in Z_{\mathcal{F}}$ be arbitrary and by induction, for every $n \in \omega$ choose a point $x_{n+1} \in Z_{\mathcal{F}} \setminus \bigcup_{i \le n}({\uparrow}x_i \cup {\downarrow}x_i) \in \mathcal{F}$. Then the injective sequence $(x_n)_{n\in\omega}$ has the required property: $x_n x_m \notin \{x_n, x_m\}$ for $n < m$ (which follows from $x_m \notin {\downarrow}x_n \cup {\uparrow}x_n$). □

A subset A of a semigroup X is called an *antichain* if $ab \notin \{a, b\}$ for any distinct points $a, b \in A$. Theorem implies the following characterization:

Corollary 1.5. *For a semilattice X the semigroup $\beta(X)$ is a band if and only if each antichain in X is finite.*

2. Semilattices whose extensions are commutative

In this section we recognize the structure of semilattices X whose extensions $\upsilon(X)$, $N_2(X)$ or $\lambda(X)$ are commutative.

Commutative semigroups of ultrafilters were characterized in [9, 4.27] as follows:

Theorem 2.1. *The Stone-Čech extension $\beta(X)$ of a semigroup X is not commutative if and only if there are sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ in X such that $\{x_k y_n : k < n\} \cap \{y_k x_n : k < n\} = \emptyset$.*

This characterization implies the following (well-known) fact:

Corollary 2.2. *If the Stone-Čech extension $\beta(X)$ of a semigroup X is commutative, then each linear subsemigroup in X in finite.*

Proof. Assume conversely that X contains an infinite linear subsemilattice L. Using Ramsey's Theorem, we can find an injective sequence $(z_n)_{n\in\omega}$ in L such that either $z_n z_m = z_n$ for all $n < m$ or else $z_n z_m = z_m$ for all $n < m$. Put $x_n = z_{2n}$ and $y_n = z_{2n+1}$ for $n \in \omega$. Applying Theorem 2.1 to the sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ we conclude that the semigroup $\beta(L)$ is not commutative. Then $\beta(X)$ is not commutative either. □

In spite of Theorem 2.1 the following problem seems to be open.

Problem 2.3. *Describe the structure of a semilattice X whose Stone-Čech extension $\beta(X)$ is commutative.*

A similar problem on commutativity of semigroups $\upsilon(X)$ also is open:

Problem 2.4. *Characterize semigroups X whose extension $\upsilon(X)$ is commutative.*

We shall resolve this problem for bands. First we prove a useful result on multiplication of upfamilies on linear semigroups.

For a semigroup X denote by $\upsilon^\bullet(X)$ the subsemigroup of $\upsilon(X)$ consisting of all upfamilies $\mathcal{A} \in \upsilon(X)$ such that for each set $A \in \mathcal{A}$ there is a finite subset $F \in \mathcal{A}$ with $F \subset A$.

For a semigroup X and two upfamilies $\mathcal{A}, \mathcal{B} \in \upsilon(X)$ let

$$\mathcal{A} \otimes \mathcal{B} = \langle A * B : A \in \mathcal{A},\ B \in \mathcal{B} \rangle.$$

It is clear that $\mathcal{A} \otimes \mathcal{B} \subset \mathcal{A} * \mathcal{B}$. In the following theorem we show that for finite linear semigroups the converse inclusion also holds.

Theorem 2.5. *If X is a linear semilattice, then $\mathcal{A} * \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$ for any upfamilies $\mathcal{A} \in \upsilon^\bullet(X)$ and $\mathcal{B} \in \upsilon(X)$.*

Proof. On the semilattice X consider the linear order $\le$ defined by: $x \le y$ iff $yx = x$. For a subset $A \subset X$ and a point $x \in X$ we write $A \le x$ if $a \le x$ for all $a \in A$. It follows from the definition of the semigroup operation $*$ on $\upsilon(X)$ that $\mathcal{A} \otimes \mathcal{B} \subset \mathcal{A} * \mathcal{B}$. To prove the reverse inclusion, fix any basic set $C = \bigcup_{a \in A} a * B_a \in \mathcal{A} * \mathcal{B}$ where $A \in \mathcal{A}$ and $B_a \in \mathcal{B}$ for all $a \in A$. Since $\mathcal{A} \in \upsilon^\bullet(X)$, we can assume that the set A is finite and hence can be enumerated as $A = \{a_1, \dots, a_n\}$ where $a_i \le a_{i+1}$ for all $i < n$. Now let us consider two cases.

1. There is $i \le n$ such that $B_{a_i} \le a_i$, which means that $a_i b = b$ for all $b \in B_{a_i}$ and hence $a_i * B_{a_i} = B_{a_i}$. For every $j \ge i$ the inequality $B_{a_i} \le a_i \le a_j$ implies $a_j * B_{a_i} = B_{a_i}$. Consequently, $A * B_{a_i} \subset \{a_1, \dots, a_{i-1}\} \cup B_{a_i}$.

We can assume that i is the smallest number such that $B_{a_i} \le a_i$. In this case the minimality of i implies that $B_{a_j} \not\le a_j$ for all $j < i$. This means $b_j \not\le a_j$ for some $b_j \in B_{a_j}$ and hence $a_j b_j = a_j$ (as $a_j b_j \in \{a_j, b_j\}$ and $a_j b_j \ne b_j$). Then $a_j * B_{a_j} \ni a_j b_j = a_j$ and thus $A * B_{a_i} \subset \{a_1, \dots, a_{i-1}\} \cup B_{a_i} \subset \bigcup_{j=1}^n a_j B_{a_j}$, which implies that $C \in \mathcal{A} \otimes \mathcal{B}$.

2. $B_{a_i} \not\le a_i$ for all $i \le n$. In this case $a_i \in a_i * B_{a_i}$ for all i. Observe that for any $b \in B_{a_n}$ and $i \le n$ we get $a_i b \in \{a_i, b\}$ by the linearity of X. If $a_i b \ne a_i$, then $a_i b = b$ and $a_i b = b = a_n a_i b = a_n b \in a_n B_{a_n}$. So,

$$\mathcal{A} \otimes \mathcal{B} \ni A * B_{a_n} \subset \{a_1, \dots, a_n\} \cup a_n B_{a_n} \subset \bigcup_{i=1}^n a_i B_{a_i} = C$$

and hence $C \in \mathcal{A} \otimes \mathcal{B}$. □

Now we are able to characterize bands X with commutative extensions $\upsilon(X)$ and $N_2(X)$.

Theorem 2.6. *For a band X the following conditions are equivalent:*

(1) *X is a finite linear semilattice;*

(2) *the semigroup $\upsilon(X)$ is commutative;*

(3) *the semigroup $N_2(X)$ is commutative;*

(4) *the semigroup $\lambda(X)$ is commutative and $(1,2)$-Clifford.*

Proof. The implication (1) ⇒ (2) follows from Theorem 2.5 as $\mathcal{A}*\mathcal{B} = \mathcal{A}\otimes\mathcal{B} = \mathcal{B}\otimes\mathcal{A} = \mathcal{B}*\mathcal{A}$ for every $\mathcal{A},\mathcal{B}\in\upsilon^\bullet(X) = \upsilon(X)$.

The implication (2) ⇒ (3) is trivial.

(3) ⇒ (1) Assume that the semigroup $N_2(X)$ is commutative. Then so is the semigroup X. Being a commutative band, the semigroup X is a semilattice. Assuming that X is not linear, we can find two points $x,y\in X$ with $xy\notin\{x,y\}$. It can be shown that the linked upfamilies $\mathcal{A} = \langle\{x,y\}\rangle$ and $\mathcal{B} = \langle\{x,xy\},\{y,xy\}\rangle\in N_2(X)$ do not commute because $\{xy\}\in\mathcal{A}*\mathcal{B}\setminus\mathcal{B}*\mathcal{A}$. Therefore, X is a linear semilattice. Since $\beta(X)\subset\upsilon(X)$ is commutative, Corollary 2.2 implies that the linear semilattice X is finite.

(1) ⇔ (4) If X is a finite linear semilattice, then $\lambda(X)$ is commutative by the implication (1) ⇒ (2) of this theorem and is $(1,2)$-Clifford by Theorem 1.3.

If the semigroup $\lambda(X)$ is commutative and $(1,2)$-Clifford, then the semigroup $X\subset\lambda(X)$ is commutative and by Theorem 1.3, X is linear. By Corollary 2.2, the linear semilattice X is finite. □

Now we shall characterize semilattices X with commutative superextension $\lambda(X)$. A semilattice X is called a *bush* if for any maximal linear subsemilattices $A,B\subset X$ the product $A*B$ is the singleton $\{\min X\}$ containing the smallest element $\min X$ of X. This definition implies that $A\cap B = A*B = \{\min X\}$. By a *branch* of a bush X we understand a maximal linear subsemilattice of X.

Theorem 2.7. *A semilattice X has commutative superextension $\lambda(X)$ if and only if X is a bush with finite branches.*

Proof. First assume that X is a bush with finite branches, and take any two maximal linked systems $\mathcal{A},\mathcal{B}\in\lambda(X)$. Since the products $\mathcal{A}*\mathcal{B}$ and $\mathcal{B}*\mathcal{A}$ are maximal linked upfamilies, the equality $\mathcal{A}*\mathcal{B} = \mathcal{B}*\mathcal{A}$ will follow as soon as we check that any two basic sets $C_{AB} = \bigcup_{a\in A} a*B_a \in \mathcal{A}*\mathcal{B}$

and $C_{BA} = \bigcup_{b\in B} b*A_b \in \mathcal{B}*\mathcal{A}$ have non-empty intersection. Here $A \in \mathcal{A}$, $(B_a)_{a\in A} \in \mathcal{B}^A$, $B \in \mathcal{B}$, and $(A_b)_{b\in B} \in \mathcal{A}^B$. Assume conversely that $C_{AB} \cap C_{BA} = \emptyset$. Then either $\min X \notin C_{AB}$ or $\min X \notin C_{BA}$.

Without loss of generality, $\min X \notin C_{AB}$. Then $\min X \notin A$ and for each $a \in A$ the set $\{a\} \cup B_a$ lies in a branch of X. Since branches of X meet only at the point $\min X$, all the sets $\{a\} \cup B_a$, $a \in A$, lie in the same (finite) branch. Repeating the argument of Theorem 2.5, we can show that $C_{AB} \supset AB'$ for some set $B' \in \mathcal{B}$. Since $\mathcal{B}$ is linked, there is a point $b \in B \cap B'$. By the same reason, there is a point $a \in A \cap A_b$. Then $ab = ba \in AB' \cap bA_b \subset C_{AB} \cap C_{BA}$ and we are done.

Now assume that X is a semilattice with commutative superextension $\lambda(X)$. Corollary 2.2 implies that all branches of X are finite. We claim that for every $z \in X$ the lower set ${\downarrow}z = \{x \in X : xz = x\}$ is linear. Assuming the converse, find two points $x, y \in {\downarrow}z$ such that $xy \notin \{x, y\}$. It follows that the points x, y, z, xy are pairwise distinct. It is easy to check that the maximal linked upfamilies $\mathcal{A} = \langle\{x, y\}, \{x, z\}, \{y, z\}\rangle$ and $\mathcal{B} = \langle\{x, y\}, \{x, xy\}, \{y, xy\}\rangle$ do not commute because $\{x, y\} \in \mathcal{B}*\mathcal{A}\setminus\mathcal{A}*\mathcal{B}$. Thus ${\downarrow}z$ is linear for every $z \in X$, which means that X is a tree.

Assuming that the tree X is not a bush, we can find two points $x, y \in X$ such that $xy \notin \{x, y, z\}$ where $z = \min X$. Now consider the maximal linked systems $\mathcal{A} = \langle\{x, y\}, \{x, z\}, \{y, z\}\rangle$, $\mathcal{B} = \langle\{x, y\}, \{x, xy\}, \{y, xy\}\rangle$ and observe that they do not commute as $\{xy\} \in \mathcal{A}*\mathcal{B}$ misses the set $\{x, y, z\} \in \mathcal{B}*\mathcal{A}$. □

3. Semigroups whose extensions are semilattices

In this section we shall characterize semigroups X whose extensions $\upsilon(X)$, $\lambda(X)$, $\varphi(X)$, or $N_2(X)$ are semilattices.

Theorem 3.1. *For a semigroup X the following conditions are equivalent:*

(1) *X is a finite linear semilattice;*

(2) *$\upsilon(X)$ is a semilattice;*

(3) *$\lambda(X)$ is a semilattice;*

(4) *$\varphi(X)$ is a semilattice.*

Proof. (1) ⇒ (2) If X is a finite linear semilattice, then $\upsilon(X)$ is a semilattice (=commutative band) by Theorems 1.1 and 2.6.

The implications (2) ⇒ (3, 4) are trivial.

The implication (3) ⇒ (1) follows from Theorems 1.1 and 2.7.

(4) ⇒ (1) Assume that $\varphi(X)$ is a semilattice. Then X, being a subsemigroup of the commutative semigroup $\varphi(X)$ is commutative. Since $\varphi(X)$ is a band, X is a linear semigroup by Theorem 1.1. Thus X, being a commutative linear semigroup, is a linear semilattice. Since the subsemigroup $\beta(X) \subset \lambda(X)$ is commutative, the linear semilattice X is finite by Corollary 2.2. □

4. Semigroups whose extensions are linear

In this section we characterize semigroups X whose extensions $\upsilon(X)$, $\lambda(X)$ or $\varphi(X)$ are linear semigroups.

A semigroup S is called a *semigroup of left (right) zeros* if $xy = x$ (resp. $xy = y$) for all $x, y \in X$.

Theorem 4.1. *For a semigroup X the semigroup $\upsilon(X)$ is linear if and only if X is either a semigroup of right zeros or a semigroup of left zeros.*

Proof. If X is a semigroup of left zeros, then for any upfamilies $\mathcal{A}, \mathcal{B} \in \upsilon(X)$ and any basic element $\bigcup_{x\in A} xB_x \in \mathcal{A} * \mathcal{B}$ we get $\bigcup_{x\in A} xB_x = \bigcup_{x\in A}\{x\} = A$ and thus $\mathcal{A} * \mathcal{B} \subset \mathcal{A}$. On the other hand, each $A \in \mathcal{A}$ belongs to $\mathcal{A} * \mathcal{B}$ as $A = A * B \in \mathcal{A} * \mathcal{B}$ for any $B \in \mathcal{B}$. Thus $\mathcal{A} * \mathcal{B} = \mathcal{A}$ and the semigroup $\upsilon(X)$ is a semigroup of left zeros.

If X is a semigroup of right zeros, then so is the semigroup $\upsilon(X)$. In both cases, $\upsilon(X)$ is linear.

Now assume that the semigroup $\upsilon(X)$ is linear. Then X, being a subsemigroup of $\upsilon(X)$, also is linear. Let x, y be any two distinct elements of X. First we prove that $xy \neq yx$. Assume conversely that $xy = yx$. Then $xy = yx \in \{x, y\}$ and we lose no generality assuming that $xy = x$. Now consider two upfamilies $\mathcal{A} = \langle\{x, y\}\rangle$ and $\mathcal{B} = \langle\{x\}, \{y\}\rangle$ and observe that

$$\mathcal{B} * \mathcal{A} = \langle\{xx, xy\}, \{yx, yy\}\rangle = \langle\{x\}, \{x, y\}\rangle = \langle\{x\}\rangle \notin \{\mathcal{A}, \mathcal{B}\},$$

so $\upsilon(X)$ is not linear and this is a required contradiction.

Thus $xy \neq yx$ for all distinct points $x, y \in X$. We call a pair $(x, y) \in X^2$ *left* if $xy = x$ and $yx = y$ and *right* if $xy = y$ and $yx = x$. Since X is linear, each pair $(x, y) \in X^2$ is either left or right. We claim that either all pairs $(x, y) \in X^2$ are left or else all such pairs are right. Assuming the opposite, find pairs $(x, y), (a, b) \in X^2$ such that (x, y) is not left and (a, b) is not right. Then $x \neq y$, $a \neq b$ and the pair (x, y) is right while (a, b) is left. Consider the filters $\mathcal{A} = \langle\{x, a\}\rangle$ and $\mathcal{B} = \langle\{y, b\}\rangle$ and observe that

$\mathcal{A} * \mathcal{B} = \langle\{xy, xb, ay, ab\}\rangle = \langle\{y, xb, ay, a\}\rangle$. Since $\upsilon(X)$ is linear, either $\mathcal{A} * \mathcal{B} = \mathcal{A}$ or $\mathcal{A} * \mathcal{B} = \mathcal{B}$. In the first case $\{x, a\} \supset \{y, xb, ay, a\} \supset \{y, a\}$ and hence $y = a$. In the second case, $\{y, a\} \subset \{y, b\}$ and thus $a = y$. Now consider the filters $\mathcal{C} = \langle\{x, b\}\rangle$ and $\mathcal{D} = \langle\{a\}\rangle$ and observe that $\mathcal{C} * \mathcal{D} = \langle\{xa, ba\}\rangle = \langle\{xy, b\}\rangle = \langle\{y, b\}\rangle = \langle\{a, b\}\rangle \notin \{\mathcal{C}, \mathcal{D}\}$, which contradicts the linearity of $\upsilon(X)$.

Therefore either each pair $(x, y) \in X^2$ is left and then X is a semigroup of left zeros or else each pair $(x, y) \in X^2$ is right and then X is a semigroup of right zeros. □

Theorem 4.2. *For a semigroup X the following conditions are equivalent:*

(1) *the semigroup $\varphi(X)$ is linear;*

(2) *the semigroup $N_2(X)$ is linear;*

(3) *either X is a semigroup of left zeros or X is a semigroup of right zeros or else X is a semilattice of order $|X| \leq 2$.*

Proof. (3) ⇒ (2) If $|X| = 1$, then $N_2(X)$ is a singleton and hence is a linear semigroup. If X is a semilattice of order $|X| = 2$, then $X = \{0, 1\}$ for some elements $0, 1$ with $0 \cdot 1 = 1 \cdot 0 = 0$. In this case $N_2(X) = \varphi(X)$ is a 3-element linear semilattice ordered as:

$$\langle\{0\}\rangle \leq \langle\{0, 1\}\rangle \leq \langle\{1\}\rangle.$$

If X is a semigroup of left or right zeros, then the semigroup $\upsilon(X)$ is linear by Theorem 4.1 and so is its subsemigroup $N_2(X)$.

(2) ⇒ (1) Is the semigroup $N_2(X)$ is linear, then so is its subsemigroup $\varphi(X)$.

(1) ⇒ (3) Assume that the semigroup $\varphi(X)$ is linear. Then X, being a subsemigroup of $\varphi(X)$, is linear as well. If $|X| \leq 2$, then either X is a linear semilattice or a semigroup of left or right zeros. So, we assume that $|X| \geq 3$. We claim that distinct elements $x, y \in X$ do not commute. Assume conversely that $xy = yx$ for some distinct elements $x, y \in X$. Since $xy = yx \in \{x, y\}$ we lose no generality assuming that $xy = yx = x$. Fix any element $z \in X \setminus \{x, y\}$. Now consider 3 cases:

1. $zx = z$. In this case we can consider the filters $\mathcal{A} = \langle\{z, y\}\rangle$ and $\mathcal{B} = \langle\{x, y\}\rangle$ and observe that

$$\mathcal{A} * \mathcal{B} = \langle\{zx, yx, zy, yy\}\rangle = \langle\{z, x, zy, y\}\rangle \notin \{\mathcal{A}, \mathcal{B}\},$$

which contradicts the linearity of $\varphi(X)$.

2. $zx = x$ and $zy = z$. In this case we can consider the filters $\mathcal{A} = \langle\{z,y\}\rangle$ and $\mathcal{B} = \langle\{x,y\}\rangle$ and observe that $\mathcal{A}*\mathcal{B} = \langle\{zx,yx,zy,yy\}\rangle = \langle\{x,x,z,y\}\rangle \notin \{\mathcal{A},\mathcal{B}\}$, which contradicts the linearity of $\varphi(X)$.

3. $zx = x$ and $zy = y$. In this case we can consider the filters $\mathcal{A} = \langle\{x,z\}\rangle$ and $\mathcal{B} = \langle\{y,z\}\rangle$ and observe that $\mathcal{A}*\mathcal{B} = \langle\{\{xy,xz,zy,zz\}\}\rangle = \langle\{x,xz,y,z\}\rangle \notin \{\mathcal{A},\mathcal{B}\}$, which again contradicts the linearity of $\varphi(X)$.

Those contradictions show that distinct elements of X do not commute. Continuing as in the proof of Theorem 4.1, we can show that X is a semigroup of right or left zeros. □

Finally, we characterize commutative semigroups with linear superextensions.

Theorem 4.3. *For a commutative semigroup X the semigroup $\lambda(X)$ is linear if and only if X is a linear semilattice of order $|X| \le 3$.*

Proof. If X is a linear semilattice of order $|X| \le 2$, then the semigroup $\lambda(X) = X$ is linear.

If X is a linear semilattice of order $|X| = 3$, then X can be identified with the set $3 = \{0,1,2\}$ endowed with the operation $xy = \min\{x,y\}$. The semigroup $\lambda(X)$ contains 4 elements: $0,1,2$ and $\Delta = \{A \subset 3 : |A| \ge 2\}$. One can check that $\lambda(3)$ is a linear semilattice ordered as follows:

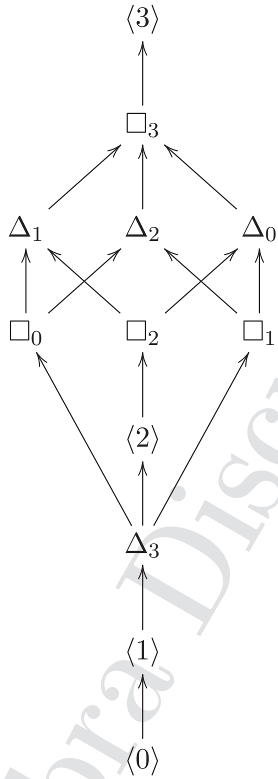
$$0 \le \Delta \le 1 \le 2.$$

This proves the "if" part of the theorem. To prove the "only if" part, we first shall analyze the structure of the superextension $\lambda(4)$ of the semilattice $4 = \{0,1,2,3\}$ endowed with the operation $xy = \min\{x,y\}$. By Theorem 3.1, $\lambda(4)$ is a semilattice. It contains 12 elements:

$$\langle k\rangle,\ \ \Delta_k = \langle\{A \subset n : |A| = 2,\ k \notin A\}\ \ \text{and}$$

$$\square_k = \langle\{n \setminus \{k\}, A : A \subset n,\ |A| = 2,\ k \in A\}\rangle\ \ \text{where } k \in 4.$$

The order structure of the semilattice $\lambda(4)$ is described in the following diagram:



Looking at this diagram we see that the semilattice $\lambda(4)$ is not linear.

Now assume that X is a commutative semigroup whose superextension $\lambda(X)$ is linear. Then X is a linear semilattice. If $|X| > 3$, then $\lambda(X)$ is not linear as it contains a subsemigroup isomorphic to the semilattice $\lambda(4)$, which is not linear. □

5. Lattices whose extensions are lattices

In this section we characterize lattices whose extensions $\upsilon(X)$, $\lambda(X)$ or $\varphi(X)$ are lattices.

A *lattice* is a set X endowed with two semilattice operations $\wedge, \vee : X \times X \to X$ such that $(x \wedge y) \vee y = y$ and $(x \vee y) \wedge y = y$ for all $x, y \in X$.

Both operations $\wedge$ and $\vee$ of a lattice X can be extended to right-topological operations $\wedge$ and $\vee$ on the compact Hausdorff space $\upsilon(X)$. Is it natural to ask if the triple $(\upsilon(X), \wedge, \vee)$ is a lattice.

A lattice will be called *linear* if $x \wedge y, x \vee y \in \{x, y\}$ for all $x, y \in X$.

Theorem 5.1. *For a lattice X the following conditions are equivalent:*

(1) *X is a linear lattice of order $|X| \le 2$.*

(2) *$\upsilon(X)$ is a lattice;*

(3) *$\lambda(X)$ is a lattice;*

(4) *$\varphi(X)$ is a lattice.*

Proof. (1) $\Rightarrow$ (2) If X is a linear lattice of order $|X| = 1$, then $\upsilon(X) = X$ is a trivial lattice. If X is a linear lattice of order 2, then X can be identified with the lattice $2 = \{0, 1\}$ endowed with the operations $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. In this case $\lambda(2) = \beta(2)$ coincides with the lattice 2, $\varphi(2) = \{\langle\{0\}\rangle, \langle\{0, 1\}\rangle, \langle\{1\}\rangle\}$ is a 3-element lattice, isomorphic to the lattice $3 = \{0, 1, 2\}$ endowed with the operations min and max, and $\upsilon(2) = \big\{\langle\{0\}\rangle, \langle\{0, 1\}\rangle, \langle\{0\}, \{1\}\rangle, \langle\{1\}\rangle\big\}$ is a 4-element lattice isomorphic to the lattice $\{0, 1\}^2$.

The implications (2) $\Rightarrow$ (3, 4) are trivial.

(3, 4) $\Rightarrow$ (1) Assume that $\lambda(X)$ or $\varphi(X)$ is a lattice. By Theorem 3.1, the lattice X is finite and linear. We claim that $|X| \le 2$. Assuming the converse, we conclude that the lattice X contains a sublattice isomorphic to the lattice $(3, \min, \max)$.

Consider the maximal linked upfamily $\Delta = \{A \subset 3 : |A| \ge 2\}$ and observe that $\max\{\Delta, \langle 1\rangle\} = \langle 1\rangle = \min\{\Delta, \langle 1\rangle\}$, which implies that $\lambda(3)$ is not a lattice and then $\lambda(X)$ also is not a lattice.

Next, consider the filters $\mathcal{A} = \langle\{0, 1, 2\}\rangle$ and $\mathcal{B} = \langle\{0, 2\}\rangle$ and observe that $\max\{\mathcal{A}, \mathcal{B}\} = \mathcal{A} = \min\{\mathcal{A}, \mathcal{B}\}$ implying that $\varphi(3)$ is not a lattice and then $\varphi(X)$ also cannot be a lattice. □

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Contact information

T. Banakh Ivan Franko National University of Lviv, Ukraine and Jan Kochanowski University, Kielce, Poland
E-Mail: t.o.banakh@gmail.com
URL: http://www.franko.lviv.ua/faculty/mechmat/Departments/Topology/bancv.html

V. Gavrylkiv Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine
E-Mail: vgavrylkiv@yahoo.com
URL: http://gavrylkiv.pu.if.ua

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